AN ANALOGUE OF THE SPACE OF CONFORMAL BLOCKS IN $(4k+2)$-DIMENSIONS

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Abstract. Based on projective representations of smooth Deligne cohomology groups, we introduce an analogue of the space of conformal blocks to compact oriented $(4k+2)$-dimensional Riemannian manifolds with boundary. For the standard $(4k+2)$-dimensional disk, we compute the space concretely to prove that its dimension is finite.

1. Introduction

As a fundamental ingredient, the *space of conformal blocks* (or the space of vacua) in the Wess-Zumino-Witten model has been investigated by many physicists and mathematicians. While its construction usually appeals to representations of affine Lie algebras [12, 13], the formulation by means of representations of loop groups ([2, 11, 15]) provides schemes for generalizations.

The theme of the present paper is an analogue of the space of conformal blocks in $(4k+2)$-dimensions. The idea of introducing such an analogue is to utilize *smooth Deligne cohomology groups* ([1, 4, 5]), or the groups of *differential characters* ([3]), instead of loop groups. In [7, 8], some properties of smooth Deligne cohomology groups, such as projective representations, are studied. In a recent work of Freed, Moore and Segal [6], similar representations are also studied in a context of *chiral (or self-dual) $2k$-forms* ([15]) on $(4k+2)$-dimensional spacetimes.

Our analogue of the space of conformal blocks is a vector space $\mathbb{V}(W, \lambda)$ associated to a compact oriented $(4k+2)$-dimensional Riemannian manifold with boundary and an element λ in a finite set $\Lambda(\partial W)$. The finite set $\Lambda(\partial W)$ is the set of equivalence classes of irreducible *admissible representations* ([8]) of the smooth Deligne cohomology group $\mathcal{G}(\partial W) = H^{2k+1}(\partial W, \mathbb{Z}(2k+1)^\infty_D)$. As will be detailed in the body of this paper (Section 2), $\mathbb{V}(W, \lambda)$ consists roughly of (dual) vectors in an irreducible representation realizing λ which are invariant under actions of *chiral (or self-dual) $2k$-forms*([6, 14]) on W.

In the case of $k = 0$, we can interpret $\mathbb{V}(W, \lambda)$ as the space of conformal blocks (or modular functor [11]) based on representations of abelian loop groups. For example, we take W to be the 2-dimensional disk $W = D^2$. In this case, $\mathcal{G}(S^1) = H^1(S^1, \mathbb{Z}(1)^\infty_D)$ is isomorphic to the loop group $LU(1)$. Irreducible admissible representations give rise to irreducible *positive energy representations* ([10]) of the loop group $LU(1)$ of level 2, which are classified by $\Lambda(S^1) \cong \mathbb{Z}_2$. Then the definition of $\mathbb{V}(D^2, \lambda)$ can be read as:

$$\mathbb{V}(D^2, \lambda) = \{\psi : \mathcal{H}_\lambda \to \mathbb{C}|\ \text{invariant under } \mathrm{Hol}(D^2, \mathbb{C}/\mathbb{Z})\},$$

where $\mathcal{H}_\lambda$ is an irreducible representation corresponding to λ on which the group $\mathrm{Hol}(D^2, \mathbb{C}/\mathbb{Z})$ of holomorphic maps $f : D^2 \to \mathbb{C}/\mathbb{Z}$ acts densely and linearly through the "Segal-Witten reciprocity law" [2, 11, 15].

A property generally desired for $\mathbb{V}(W, \lambda)$ is its finite-dimensionality. In the case of $k = 0$, there is a result of Segal regarding the property [11]. The purpose of this paper is to prove that $\mathbb{V}(W, \lambda)$ is finite-dimensional at least in the case where W is the $(4k + 2)$-dimensional disk $D^{4k+2} = \{x \in \mathbb{R}^{4k+2} |\ |x| \leq 1\}$. Note that we have $\Lambda(S^{4k+1}) = \{0\}$ for $k > 0$. Then we can show:

Theorem 1.1. *If $k > 0$, then $\mathbb{V}(D^{4k+2}, 0) \cong \mathbb{C}$.*

The essential part of the proof is a fact about chiral $2k$-forms on D^{4k+2}, which we derive from [9]. (See Section 3 for detail.) The proof of Theorem 1.1 is applicable to the case of $k = 0$, and we have:

$$\mathbb{V}(D^2, \lambda) \cong \begin{cases} \mathbb{C}, & (\lambda = 0) \\ 0. & (\lambda = 1) \end{cases}$$

This result is consistent with the known fact about the dimension of the space of conformal blocks in the $U(1)$ Wess-Zumino-Witten model at level 2 ([11, 13]).

The finite-dimensionality of $\mathbb{V}(W, \lambda)$ for general W remains open at present. A possible approach toward the issue is to generalize Segal's idea (p.431, [11]), which should be examined in future studies.

2. Analogue of the space of conformal blocks

In this section, we introduce the vector space $\mathbb{V}(W, \lambda)$. For this aim, we summarize some results in [7, 8]. In particular, we review central extensions of smooth Deligne cohomology groups, a generalization of the Segal-Witten reciprocity law, and admissible representations.

2.1. Central extension.

To begin with, we recall the definition of *smooth Deligne cohomology* [1, 4, 5]. For a non-negative integer p and a smooth manifold X, the *(complexified) smooth Deligne cohomology group* $H^*(X, \mathbb{Z}(p)^\infty_{D,\mathbb{C}})$ is defined to be the hypercohomology of the following complex of sheaves on X:

$$\mathbb{Z}(q)^\infty_{D,\mathbb{C}} : \ \mathbb{Z} \longrightarrow \underline{A}^0_{\mathbb{C}} \xrightarrow{d} \underline{A}^1_{\mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^{q-1}_{\mathbb{C}} \longrightarrow 0 \longrightarrow \cdots ,$$

where $\mathbb{Z}$ is the constant sheaf, and $\underline{A}^q_{\mathbb{C}}$ the sheaf of germs of $\mathbb{C}$-valued q-forms. We fix a non-negative integer k, and put $\mathcal{G}(X)_{\mathbb{C}} = H^{2k+1}(X, \mathbb{Z}(2k + 1)^\infty_{D,\mathbb{C}})$ for a smooth manifold X.

Proposition 2.1 ([7])**.** *For a compact oriented $(4k+1)$-dimensional smooth manifold M without boundary, there is a non-trivial central extension $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$ of $\mathcal{G}(M)_{\mathbb{C}}$:*

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \tilde{\mathcal{G}}(M)_{\mathbb{C}} \longrightarrow \mathcal{G}(M)_{\mathbb{C}} \longrightarrow 1.$$

The central extension $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$ is induced from the group 2-cocycle $S_{M,\mathbb{C}} : \mathcal{G}(M)_{\mathbb{C}} \times \mathcal{G}(M)_{\mathbb{C}} \to \mathbb{C}/\mathbb{Z}$ defined by $S_{M,\mathbb{C}}(f, g) = \int_M f \cup g$, where $\int_M$ and $\cup$ are the cup product and the integration in smooth Deligne cohomology.

For a smooth manifold X, the smooth Deligne cohomology $H^1(X, \mathbb{Z}(1)^\infty_{D,\mathbb{C}})$ is naturally isomorphic to $C^\infty(X, \mathbb{C}/\mathbb{Z})$. Thus, if $k = 0$ and $M = S^1$, then we can identify $\mathcal{G}(S^1)_{\mathbb{C}}$ with the loop group $L\mathbb{C}^*$. In this case, $\tilde{\mathcal{G}}(S^1)_{\mathbb{C}}$ is isomorphic to $\widehat{L\mathbb{C}^*}/\mathbb{Z}_2$, where $\widehat{L\mathbb{C}^*}$ is the *universal central extension* of $L\mathbb{C}^*$, ([10]).

2.2. **A generalization of the Segal-Witten reciprocity law.** Let W be a compact oriented $(4k+2)$-dimensional Riemannian manifold W possibly with boundary. We denote by $A^{2k+1}(W,\mathbb{C})$ the space of $\mathbb{C}$-valued $(2k+1)$-forms on W. The Hodge star operator $*: A^{2k+1}(W,\mathbb{C}) \to A^{2k+1}(W,\mathbb{C})$ satisfies $** = -1$. Notice that, in general, the smooth Deligne cohomology $\mathcal{G}(X_{\mathbb{C}}) = H^{2k+1}(X, \mathbb{Z}(2k+1)^{\infty}_{D,\mathbb{C}})$ fits into the following exact sequence.

$$0 \to H^{2k}(W,\mathbb{C}/\mathbb{Z}) \to \mathcal{G}(W)_{\mathbb{C}} \xrightarrow{\delta} A^{2k+1}(W,\mathbb{C})_{\mathbb{Z}} \to 0,$$

where $A^{2k+1}(W,\mathbb{C})_{\mathbb{Z}} \subset A^{2k+1}(W,\mathbb{C})$ is the subgroup consisting of closed integral forms. Using $*$ and δ, we define the subgroups $\mathcal{G}(W)^{\pm}_{\mathbb{C}}$ in $\mathcal{G}(W)_{\mathbb{C}}$ by

$$\mathcal{G}(W)^{\pm}_{\mathbb{C}} = \{f \in \mathcal{G}(W)_{\mathbb{C}} |\ \delta(f) \mp \sqrt{-1} * \delta(f) = 0\}.$$

We call $\mathcal{G}(W)^{+}_{\mathbb{C}}$ the *chiral subgroup*, since $2k$-forms $B \in A^{2k}(W,\mathbb{C})$ such that $dB = i * dB$ are called *chiral (or self-dual) $2k$-forms.* (See [6, 14] for example.)

Proposition 2.2 ([7])**.** *For a compact oriented $(4k+2)$-dimensional Riemannian manifold W with boundary, the following map is a homomorphism:*

$$\tilde{r}^{+} : \mathcal{G}(W)^{+}_{\mathbb{C}} \longrightarrow \tilde{\mathcal{G}}(\partial W)_{\mathbb{C}}, \quad f \mapsto (f|_{\partial W}, 1).$$

In the case of $k=0$, W is a Riemann surface. Since $\mathcal{G}(W)^{+}_{\mathbb{C}}$ is identified with the group of holomorphic functions $f : W \to \mathbb{C}/\mathbb{Z}$, Proposition 2.2 recovers the "Segal-Witten reciprocity law"([2, 11, 15]) for $\widehat{L\mathbb{C}^*}/\mathbb{Z}_2$.

2.3. **Admissible representations.** The group $\mathcal{G}(X)_{\mathbb{C}} = H^{2k+1}(X, \mathbb{Z}(2k+1)^{\infty}_{D,\mathbb{C}})$ can be thought of as a complexification of the (real) smooth Deligne cohomology $\mathcal{G}(X) = H^{2k+1}(X, \mathbb{Z}(2k+1)^{\infty}_{D})$ defined as the hypercohomology of the following complex of sheaves:

$$\mathbb{Z}(2k+1)^{\infty}_{D} :\ \mathbb{Z} \longrightarrow \underline{A}^0 \xrightarrow{d} \underline{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^{2k} \longrightarrow 0 \longrightarrow \cdots,$$

where $\underline{A}^q$ is the sheaf of germs of $\mathbb{R}$-valued q-forms.

For a compact oriented $(4k+1)$-dimensional Riemannian manifold M without boundary, *admissible representations* of $\mathcal{G}(M)$ are introduced in [8]. An admissible representation $\rho : \mathcal{G}(M) \times \mathcal{H} \to \mathcal{H}$ of $\mathcal{G}(M)$ is a certain projective representation on a Hilbert space $\mathcal{H}$, and gives a linear representation $\tilde{\rho} : \tilde{\mathcal{G}}(M) \times \mathcal{H} \to \mathcal{H}$ of the central extension $\tilde{\mathcal{G}}(M)$ induced from the natural inclusion $\mathcal{G}(M) \subset \mathcal{G}(M)_{\mathbb{C}}$:

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] \arrow[d] & \tilde{\mathcal{G}}(M) \arrow[r] \arrow[d] & \mathcal{G}(M) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \tilde{\mathcal{G}}(M)_{\mathbb{C}} \arrow[r] & \mathcal{G}(M)_{\mathbb{C}} \arrow[r] & 1.
\end{tikzcd}$$

The set $\Lambda(M)$ of equivalence classes of irreducible admissible representations of $\mathcal{G}(M)$ is a finite set [8]. For example, if $H^{2k+1}(M,\mathbb{Z})$ is torsion free, then we can identify $\Lambda(M)$ with $H^{2k+1}(M,\mathbb{Z}_2)$. We write $(\tilde{\rho}_\lambda, \mathcal{H}_\lambda)$ for the linear representation of $\tilde{\mathcal{G}}(M)$ realizing $\lambda \in \Lambda$.

Proposition 2.3 ([8])**.** *Let M be a compact oriented $(4k+1)$-dimensional Riemannian manifold M without boundary. For $\lambda \in \Lambda(M)$, there exists an invariant dense subspace $\mathcal{E}_\lambda \subset \mathcal{H}_\lambda$, and the representation $\tilde{\rho}_\lambda : \tilde{\mathcal{G}}(M) \times \mathcal{E}_\lambda \to \mathcal{E}_\lambda$ extends to a linear representation $\tilde{\rho}_\lambda : \tilde{\mathcal{G}}(M)_{\mathbb{C}} \times \mathcal{E}_\lambda \to \mathcal{E}_\lambda$ of $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$.*

We notice that $\tilde{\rho}_\lambda(f) : \mathcal{E}_\lambda \to \mathcal{E}_\lambda$ is generally unbounded, so that the action of $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$ on $\mathcal{E}_\lambda$ does not extends to the whole of $\mathcal{H}_\lambda$.

In the case of $k = 0$ and $M = S^1$, we can identify $\mathcal{G}(S^1)$ with the loop group $LU(1)$, which has $\mathcal{G}(S^1)_{\mathbb{C}} \cong L\mathbb{C}^*$ as a complexification. Admissible representations of $\mathcal{G}(S^1)$ give rise to positive energy representations of level 2. As is known [10], the equivalence classes of irreducible positive energy representations of $LU(1)$ of level 2 are in one to one correspondence with the elements in $\Lambda(S^1) \cong \mathbb{Z}_2$. A positive energy representation of $LU(1)$ extends to a representation of $L\mathbb{C}^*$ on an invariant dense subspace.

2.4. **Analogue of the space of conformal blocks.** We use Proposition 2.2 and Proposition 2.3 to formulate our analogue of the space of conformal blocks:

Definition 2.4. Let W be a compact oriented $(4k+2)$-dimensional Riemannian manifold with boundary. For $\lambda \in \Lambda(\partial W)$, we define $\mathbb{V}(W, \lambda)$ to be the vector space consisting of continuous linear maps $\psi : \mathcal{E}_\lambda \to \mathbb{C}$ invariant under the action of $\mathcal{G}(W)^+_{\mathbb{C}}$ through $\tilde{r}^+$:

$$\begin{aligned}\mathbb{V}(W, \lambda) &= \mathrm{Hom}(\mathcal{E}_\lambda, \mathbb{C})^{\mathrm{Im}\tilde{r}^+} \\ &= \{\psi : \mathcal{E}_\lambda \to \mathbb{C} |\ \psi(\tilde{\rho}_\lambda(\tilde{r}^+(f))v) = \psi(v) \text{ for } v \in \mathcal{E}_\lambda \text{ and } f \in \mathcal{G}(W)^+_{\mathbb{C}}\}.\end{aligned}$$

Since the subgroup $\mathbb{C}^*$ in $\tilde{\mathcal{G}}(M)_{\mathbb{C}} = \mathcal{G}(M)_{\mathbb{C}} \times \mathbb{C}^*$ acts on $\mathcal{E}_\lambda$ by the scalar multiplication, we can formulate $\mathbb{V}(W, \lambda)$ in terms of the projective representation $(\rho_\lambda, \mathcal{E}_\lambda)$ corresponding to $(\tilde{\rho}_\lambda, \mathcal{E}_\lambda)$:

$$\begin{aligned}\mathbb{V}(W, \lambda) &= \mathrm{Hom}(\mathcal{E}_\lambda, \mathbb{C})^{\mathrm{Im}r^+} \\ &= \{\psi : \mathcal{E}_\lambda \to \mathbb{C} |\ \psi(\rho_\lambda(r^+(f))v) = \psi(v) \text{ for } v \in \mathcal{E}_\lambda \text{ and } f \in \mathcal{G}(W)^+_{\mathbb{C}}\},\end{aligned}$$

where $r^+ : \mathcal{G}(W)^+_{\mathbb{C}} \to \mathcal{G}(\partial W)_{\mathbb{C}}$ is the restriction: $r^+(f) = f|_{\partial W}$.

Remark 1. One may wonder why we use representations of $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$ on pre-Hilbert spaces to formulate $\mathbb{V}(W, \lambda)$, instead of unitary representations of $\tilde{\mathcal{G}}(M)$ on Hilbert spaces. The reason is that we cannot introduce a counterpart of the chiral subgroup $\mathcal{G}(W)^+_{\mathbb{C}}$ to $\mathcal{G}(W)$. Notice, however, that we can formulate $\mathbb{V}(W, \lambda)$ as

$$\mathbb{V}(W, \lambda) = \{\psi : \mathcal{H}_\lambda \to \mathbb{C} |\ \psi(\rho_\lambda(r^+(f))v) = \psi(v) \text{ for } v \in \mathcal{E}_\lambda \text{ and } f \in \mathcal{G}(W)^+_{\mathbb{C}}\},$$

because $\mathcal{E}_\lambda$ is dense in $\mathcal{H}_\lambda$.

3. Calculation of $\mathbb{V}(D^{4k+2}, \lambda)$

In this section, we prove Theorem 1.1. As preparations for the proof, we review in some detail the construction of irreducible representations of Heisenberg groups in [10]. We also study chiral $2k$-forms on $\mathbb{R}^{4k+2}$ by the help of results in [9].

3.1. **Representation of Heisenberg group.** For a compact oriented $(4k+1)$-dimensional Riemannian manifold M without boundary, the group $\mathcal{G}(M)_{\mathbb{C}}$ admits the decomposition:

$$\begin{aligned}\mathcal{G}(M)_{\mathbb{C}} &\cong (A^{2k}(M, \mathbb{C})/A^{2k}(M, \mathbb{C})_{\mathbb{Z}}) \times H^{2k+1}(M, \mathbb{Z}) \\ &\cong (\mathbb{H}^{2k}(M, \mathbb{C})/\mathbb{H}^{2k}(M, \mathbb{C})_{\mathbb{Z}}) \times d^*(A^{2k+1}(M, \mathbb{C})) \times H^{2k+1}(M, \mathbb{Z}),\end{aligned}$$

where $\mathbb{H}^{2k}(M, \mathbb{C})$ is the group of $\mathbb{C}$-valued harmonic $2k$-forms, $\mathbb{H}^{2k}(M, \mathbb{C})_{\mathbb{Z}} = \mathbb{H}^{2k}(M, \mathbb{C}) \cap A^{2k}(M, \mathbb{C})_{\mathbb{Z}}$ the subgroup of integral harmonic $2k$-forms, and d^* :

$A^{2k+1}(M,\mathbb{C}) \to A^{2k}(M,\mathbb{C})$ the formal adjoint of the exterior differential. Thus, in particular, if M is such that $b^{2k+1}(M) = 0$, then $\mathcal{G}(M)_{\mathbb{C}} \cong d^*(A^{2k+1}(M,\mathbb{Z}))$. The representations $(\tilde{\rho}_\lambda, \mathcal{E}_\lambda)$ of $\tilde{\mathcal{G}}(M)_{\mathbb{C}}$ in Proposition 2.3 are built on a projective representation (ρ, E) of $d^*(A^{2k+1}(M,\mathbb{C})$. We review here the construction of (ρ, E) following [10], and give a simple consequence.

As in [8], we define the Hermitian inner product $(\ ,\ )_V$ on $d^*(A^{2k+1}(M,\mathbb{C}))$ by that induced from the Sobolev norm $\|\cdot\|_s$ with $s = 1/2$. (Our convention is that $(\ ,\ )_V$ is $\mathbb{C}$-linear in the first variable, which differs from that in [10].) On the completion $V_{\mathbb{C}}$ of $d^*(A^{2k+1}(M,\mathbb{C}))$, we define the linear map $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ by $J = \tilde{J}/|\tilde{J}|$, where $\tilde{J} : d^*(A^{2k+1}(M,\mathbb{C})) \to d^*(A^{2k+1}(M,\mathbb{C}))$ is the differential operator $\tilde{J} = *d$. Then J is a complex structure compatible with $(\ ,\ )_V$, and satisfies:

$$(\alpha, J\overline{\beta})_V = \int_M \alpha \wedge d\beta$$

for $\alpha, \beta \in d^*(A^{2k+1}(M,\mathbb{C}))$. By means of J, we decompose $V_{\mathbb{C}}$ into $V_{\mathbb{C}} = W \oplus \overline{W}$, where J acts on W and $\overline{W}$ by $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Then we let $E = \mathbb{C}\langle \epsilon_\xi |\ \xi \in W\rangle$ be the vector space generated by the symbols ϵ_ξ corresponding to $\xi \in W$, and $\langle\ ,\ \rangle : E \times E \to \mathbb{C}$ the Hermitian inner product $\langle \epsilon_\xi, \epsilon_\eta\rangle = e^{2(\xi,\eta)_V}$. For $v_+ \in W$ and $v_- \in \overline{W}$, we define $\rho(v_+ + v_-) :\ E \to E$ by

$$\rho(v_+ + v_-)\epsilon_\xi = \exp\left(-(v_+, \overline{(v_-)})_V - 2(\xi, \overline{(v_-)})_V\right)\epsilon_{\xi + v_+}.$$

We can verify $\rho(v)\rho(v')\epsilon_\xi = e^{\sqrt{-1}(v,J\bar{v}')_V}\rho(v+v')\epsilon_\xi$ for $v, v' \in V_{\mathbb{C}}$, so that we have a projective representation $\rho : V_{\mathbb{C}} \times E \to E$. Because the group 2-cocycle $S_{M,\mathbb{C}}$ on $d^*(A^{2k+1}(M,\mathbb{C}))$ has the expression :

$$S_{M,\mathbb{C}}(\alpha,\beta) = \int_M \alpha \wedge d\beta \mod \mathbb{Z},$$

we get the projective representation $\rho : d^*(A^{2k+1}(M,\mathbb{C})) \times E \to E$.

In general, $\rho(\alpha) : E \to E$ is unbounded. However, if α belongs to the real vector space $d^*(A^{2k+1}(M))$ underlying $d^*(A^{2k}(M,\mathbb{C}))$, then $\rho(\alpha) : E \to E$ is isometric. Thus, $\rho(\alpha)$ extends to a unitary map on the completion $H = \overline{E}$ of E, and we have an irreducible projective unitary representation $\rho : d^*(A^{2k+1}(M)) \times H \to H$. As is shown in [10], we can identify $\overline{E}$ with a completion of the symmetric algebra $S(W)$ by the mapping $\epsilon_\xi \mapsto e^\xi = \sum_{j=0}^\infty \xi^j/j!$.

The next lemma is a simple consequence from the above construction.

Lemma 3.1. *Let (ρ, E) be as above.*

(a) The vector space $\mathrm{Hom}(E,\mathbb{C})^W$ is generated by the continuous linear map $\chi : E \to \mathbb{C}$ defined by $\chi(v) = \langle v, \epsilon_0\rangle$:

$$\mathrm{Hom}(E,\mathbb{C})^W = \mathbb{C}\langle\chi\rangle.$$

(b) For a dense subspace U in W, we have $\mathrm{Hom}(E,\mathbb{C})^U = \mathrm{Hom}(E,\mathbb{C})^W$.

Proof. To prove (a), we begin with proving the W-invariance of χ. Notice that $\chi(\epsilon_\xi) = 1$ for all $\xi \in W$. For $f \in W$ and $v = \sum_j c_j \epsilon_{\xi_j} \in E$, we have:

$$\chi(v) = \sum_j c_j \chi(\epsilon_{\xi_j}) = \sum_j c_j,$$
$$\chi((\rho(f)v) = \sum_j c_j \chi(\rho(f)\epsilon_{\xi_j}) = \sum_j c_j \chi(\epsilon_{\xi_j + f}) = \sum_j c_j.$$

Hence χ is invariant under the action of W, and $\mathbb{C}\langle\chi\rangle \subset \mathrm{Hom}(E, \mathbb{C})^W$. To see $\mathbb{C}\langle\chi\rangle \supset \mathrm{Hom}(E, \mathbb{C})^W$, we show that any $\psi \in \mathrm{Hom}(E, \mathbb{C})^W$ is of the form $\psi = c\chi$ for some $c \in \mathbb{C}$. For $v = \sum_j c_j \epsilon_{\xi_j} \in E$, the invariance of ψ leads to:

$$\begin{aligned} \psi(v) &= \sum_j c_j \psi(\epsilon_{\xi_j}) = \sum_j c_j \psi(\rho(\xi_j)\epsilon_0) = \sum_j c_j \psi(\epsilon_0) \\ &= \psi(\epsilon_0) \sum_j c_j = \psi(\epsilon_0)\chi(v). \end{aligned}$$

If we put $c = \psi(\epsilon_0)$, then $\psi = c\chi$. For (b), it suffices to prove the inclusion $\mathrm{Hom}(E, \mathbb{C})^U \subset \mathrm{Hom}(E, \mathbb{C})^W$. So we will show $\psi \in \mathrm{Hom}(E, \mathbb{C})^U$ is also invariant under W. For $f \in W$, there is a sequence $\{f_n\}$ in U converging to f. Notice that $\rho(\cdot)v : W \to E$ is continuous for $v \in E$. Now, we have:

$$\psi(\rho(f)v) = \psi(\rho(\lim_{n\to\infty} f_n)v) = \lim_{n\to\infty} \psi(\rho(f_n)v) = \lim_{n\to\infty} \psi(v) = \psi(v),$$

so that $\psi \in \mathrm{Hom}(E, \mathbb{C})^W$. □

Remark 2. The key to Lemma 3.1 (b) is that the map $\rho(\cdot)v : W \to E$ is continuous for each $v \in W$. The representations $\tilde{\rho}_\lambda : \tilde{\mathcal{G}}(M)_{\mathbb{C}} \times \mathcal{E}_\lambda \to \mathcal{E}_\lambda$ in Proposition 2.3 have the same property [8].

3.2. **Chiral $2k$-forms on $\mathbb{R}^{4k+2}$.** The Laplacian $\Delta = dd^* + d^*d$ preserves the subspace $d^*(A^{2k+1}(S^{4k+1}, \mathbb{C}))$. For an eigenvalue ℓ of Δ, we define V_ℓ to be the following eigenspace:

$$V_\ell = \{\beta \in d^*(A^{2k+1}(S^{4k+1}, \mathbb{C}))|\ \Delta\beta = \ell\beta\}.$$

The complex structure J, introduced in the previous subsection, preserves V_ℓ. (In particular, $J = *d/\sqrt{\ell}$ on V_ℓ.) So we have the decomposition $V_\ell = W_\ell \oplus \overline{W}_\ell$, where J acts on W_ℓ and $\overline{W}_\ell$ by $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Proposition 3.2. *There is the following relation of inclusion:*

$$\bigoplus_\ell W_\ell \subset \mathrm{Im}\{i^* : A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+ \to A^{2k}(S^{4k+1}, \mathbb{C})\} \subset W,$$

where $\bigoplus$ means the algebraic direct sum, ℓ runs through all the distinct eigenvalues, $i : S^{4k+1} \to \mathbb{R}^{4k+2}$ is the inclusion, and $A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^\pm$ are the spaces of chiral and anti-chiral $2k$-forms on $\mathbb{R}^{4k+2}$:

$$A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^\pm = \{B \in A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})|\ dB \mp \sqrt{-1} * dB = 0\}.$$

For the proof, we use some results shown by Ikeda and Taniguchi in [9]. To explain the relevant results, we introduce some notations. Let $S^i(\mathbb{R}^{4k+1})$ and $\Lambda^p(\mathbb{R}^{4k+1})$ be the spaces of the symmetric tensors of degree i and anti-symmetric

tensors of degree p. We put $P_i^p = S^i(\mathbb{R}^{4k+1}) \otimes \Lambda^p(\mathbb{R}^{4k+1}) \otimes \mathbb{C}$, and regard P_i^p as a subspace in $A^p(\mathbb{R}^{4k+1}, \mathbb{C})$. We then define the vector spaces:

$$\begin{aligned} H_i^p &= \mathrm{Ker}\Delta \cap \mathrm{Ker}d^* \cap P_i^p, \\ {}'H_i^p &= \mathrm{Ker}d \cap H_i^p, \\ {}''H_i^p &= \mathrm{Ker}i\Big(r\frac{d}{dr}\Big) \cap H_i^p, \end{aligned}$$

where $i\left(r\frac{\partial}{\partial r}\right)$ is the contraction with the vector field $r\frac{d}{dr} = \sum_{j=1}^{4k+2} x_j \frac{d}{dx_j}$.

Notice that the standard action of $SO(4k+2)$ on $\mathbb{R}^{4k+2}$ makes ${}'H_i^p$ and ${}''H_i^p$ into $SO(4k+2)$-modules. Similarly, V_ℓ is also an $SO(4k+2)$-module. From [9] (Theorem 6.8, p. 537), we can derive:

Proposition 3.3 ([9])**.** *Let $\ell_1 < \ell_2 < \ell_3 < \cdots$ be the sequence of distinct eigenvalues of Δ on $d^*(A^{2k+1}(S^{4k+1}, \mathbb{C}))$. For $i \in \mathbb{N}$, we have:*

(a) The maps $i^* : A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C}) \to A^{2k}(S^{4k+1}, \mathbb{C})$ and $d : A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C}) \to A^{2k+1}(\mathbb{R}^{4k+2}, \mathbb{C})$ induce the following isomorphisms of $SO(4k+2)$-modules:

$$V_{\ell_i} \xleftarrow{\ i^*\ } {}''H_i^{2k} \xrightarrow{\ d\ } {}'H_{i-1}^{2k+1}.$$

(b) The $SO(4k+2)$-module V_{ℓ_i} decomposes into two distinct irreducible modules having the same dimensions.

Remark 3. More precisely, the sequence $\{\ell_i\}_{i\in\mathbb{N}}$ is given by $\ell_i = (2k+i)^2$, and the dimension of the two irreducible modules in V_{l_i} is $\binom{4k+i}{2k}\binom{2k+i-1}{2k}$.

We also note the next lemma for later use:

Lemma 3.4. *Let $(\ ,\ )_{L^2}$ be the L^2-norm on $A^{2k+1}(D^{4k+2}, \mathbb{C})$.*

(a) For $B, B' \in A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})$, we have:

$$(i^*B, Ji^*B')_V = -(dB, *dB')_{L^2}.$$

(b) If $B \in A^{2k+1}(\mathbb{R}^{4k+2}, \mathbb{C})$ obeys $(J - \sqrt{-1})i^*B = 0$, then:

$$\|H^+\|_{L^2}^2 - \|H^-\|_{L^2}^2 \geq 0,$$

where $H^\pm = (1 \pm \sqrt{-1}*)dB/2$. Similarly, if $(J + \sqrt{-1})i^*B = 0$, then:

$$\|H^-\|_{L^2}^2 - \|H^+\|_{L^2}^2 \geq 0.$$

Proof. We can readily show (a) combining properties of $(\ ,\ )_V$ and J with Stokes' theorem. Notice that the eigenspaces $\mathrm{Ker}(1 \pm \sqrt{-1}*)$ in $A^{2k+1}(D^{4k+2}, \mathbb{C})$ are orthogonal to each other with respect to the L^2-norm. Then the inequalities in (b) follow from $(i^*B, i^*B)_V \geq 0$ and (a). □

Proposition 3.3 and the above lemma yield:

Lemma 3.5. *The map i^* induces the following isomorphisms for $i \in \mathbb{N}$:*

$${}''H_i^{2k} \cap A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+ \cong W_{\ell_i}, \quad {}''H_i^{2k} \cap A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^- \cong \overline{W}_{\ell_i}.$$

Proof. Notice that the action of $SO(4k+2)$ on V_{ℓ_i} is compatible with J. So W_{ℓ_i} and $\overline{W}_{\ell_i}$ are $SO(4k+2)$-modules. The dimensions of W_{ℓ_i} and $\overline{W}_{\ell_i}$ are the same, since they are complex-conjugate to each other. Similarly, since the $SO(4k+2)$-action on ${}'H_{i-1}^{2k+1}$ is compatible with the Hodge star operator $*$, the vector spaces $({}'H_{i-1}^{2k+1})^\pm = {}'H_{i-1}^{2k+1} \cap \mathrm{Ker}(1 \mp \sqrt{-1}*)$ are also $SO(4k+2)$-modules with the same dimensions.

Thus, by Proposition 3.3, W_{ℓ_i} is isomorphic to one of $({}'H^{2k+1}_{i-1})^{\pm}$ through $d \circ (i^*)^{-1}$, and $\overline{W}_{\ell_i}$ is isomorphic to the other. To settle the case, we appeal to Lemma 3.4 (b). Then the case of $W_{\ell_i} \cong ({}'H^{2k+1}_{i-1})^{+}$ and $\overline{W}_{\ell_i} \cong ({}'H^{2k+1}_{i-1})^{-}$ is consistent. Now the isomorphisms $d : {}''H^{2k}_i \cap A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^{\pm} \to ({}'H^{2k+1}_{i-1})^{\pm}$ complete the proof. □

The proof of Proposition 3.2. By Lemma 3.5 we have:

$$W_{\ell_i} \subset \mathrm{Im}\{i^* : A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+ \to A^{2k}(S^{4k+1}, \mathbb{C})\},$$

which leads to the first inclusion in Proposition 3.2. For the second inclusion, we recall that the subspaces W and $\overline{W}$ in $V_{\mathbb{C}}$ are orthogonal with respect to $(\ ,\ )_V$. So, it suffices to verify the image $i^*(A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+)$ is orthogonal to $\overline{W}$. By Lemma 3.5, we also have:

$$\overline{W}_{\ell_i} \subset \mathrm{Im}\{i^* : A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^- \to A^{2k}(S^{4k+1}, \mathbb{C})\}.$$

Thus, by the help of Lemma 3.4 (a), we see that $i^*(A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+)$ is orthogonal to each $\overline{W}_{\ell_i}$. Because $\bigoplus_i W_{\ell_i}$ forms a dense subspace in W, the image $i^*(A^{2k}(\mathbb{R}^{4k+2}, \mathbb{C})^+)$ is orthogonal to $\overline{W}$. □

3.3. **Proof of the main result.** We now compute $\mathbb{V}(D^{4k+2}, \lambda)$.

First, we consider the case of $k > 0$. In this case, we have:

$$\mathcal{G}(S^{4k+1})_{\mathbb{C}} = A^{2k}(S^{4k+1}, \mathbb{C})/A^{2k}(S^{4k+1}, \mathbb{C})_{\mathbb{Z}} \cong d^*(A^{2k+1}(S^{4k+1}, \mathbb{C})).$$

So the projective unitary representation (ρ, H) reviewed in Subsection 3.1 realizes the unique element in $\Lambda(S^{4k+1}) = \{0\}$, and E gives the invariant dense subspace in Proposition 2.3.

Theorem 3.6. *If $k > 0$, then $\mathbb{V}(D^{4k+2}, 0) \cong \mathbb{C}$.*

Proof. Note that $\mathcal{G}(D^{4k+2})^+_{\mathbb{C}} = A^{2k}(D^{4k+2}, \mathbb{C})^+/A^{2k}(D^{4k+1}, \mathbb{C})_{\mathbb{Z}}$. Proposition 3.2 leads to: $U \subset \mathrm{Im} r^+ \subset W$, where the dense subspace U in W is given by $U = \bigoplus_{i \in \mathbb{N}} W_{\ell_i}$. This relation of inclusion implies:

$$\mathrm{Hom}(E, \mathbb{C})^U \supset \mathrm{Hom}(E, \mathbb{C})^{\mathrm{Im} r^+} \supset \mathrm{Hom}(E, \mathbb{C})^W.$$

Therefore Lemma 3.1 establishes the theorem. □

In the case of $k = 0$, we have the familiar decomposition of $\mathcal{G}(S^1)_{\mathbb{C}} \cong L\mathbb{C}^*$:

$$\mathcal{G}(S^1)_{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z} \times \{\phi : S^1 \to \mathbb{R} |\ \textstyle\int \phi(\theta) d\theta = 0\} \times \mathbb{Z}.$$

As is mentioned, admissible representations of $\mathcal{G}(S^1)$ are equivalent to positive energy representations of $LU(1)$ of level 2. For $\lambda \in \Lambda(S^1) = \mathbb{Z}_2 = \{0, 1\}$, the invariant dense subspace $\mathcal{E}_\lambda$ in Proposition 2.3 is given by $\mathcal{E}_\lambda = \bigoplus_{\xi \in \mathbb{Z}} E_{\lambda + 2\xi}$, where $E_{\lambda+2\xi} = E$ is the pre-Hilbert space in Subsection 3.1 and the subgroup of constant loops $\mathbb{C}/\mathbb{Z} \subset \mathcal{G}(S^1)_{\mathbb{C}}$ acts on $E_{\lambda+2\xi}$ with weight $\lambda + 2\xi$.

Proposition 3.7. *For $\lambda \in \Lambda(S^1) = \mathbb{Z}_2$, we have:*

$$\mathbb{V}(D^2, \lambda) \cong \begin{cases} \mathbb{C}, & (\lambda = 0) \\ 0. & (\lambda = 1) \end{cases}$$

Proof. Clearly, constant loops $S^1 \to \mathbb{C}/\mathbb{Z}$ extend to holomorphic maps $D^2 \to \mathbb{C}/\mathbb{Z}$. So we use Proposition 3.2 to obtain the relation of inclusion:

$$\mathbb{C}/\mathbb{Z} \times U \subset \mathrm{Im} r^+ \subset \mathbb{C}/\mathbb{Z} \times W,$$

where $U = \bigoplus_{i \in \mathbb{N}} W_{\ell_i}$. Since $\mathbb{C}/\mathbb{Z}$ acts on $E_{\lambda+2\xi}$ with weight $\lambda + 2\xi$, we have:

$$\mathrm{Hom}(\mathcal{E}_\lambda, \mathbb{C})^{\mathbb{C}/\mathbb{Z}} \subset \prod_{\xi \in \mathbb{Z}} \mathrm{Hom}(E_{\lambda+2\xi}, \mathbb{C})^{\mathbb{C}/\mathbb{Z}} \cong \begin{cases} \mathrm{Hom}(E_0, \mathbb{C}), & (\lambda = 0) \\ \{0\}. & (\lambda = 1) \end{cases}$$

Thus, if $\lambda = 1$, then $\mathbb{V}(D^2, \lambda) = \{0\}$. In the case of $\lambda = 0$, we have:

$$\mathrm{Hom}(E_0, \mathbb{C})^U \supset \mathrm{Hom}(E_0, \mathbb{C})^{\mathrm{Im} r^+} \supset \mathrm{Hom}(E_0, \mathbb{C})^W.$$

Now, Lemma 3.1 completes the proof. □

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